ME 3304 Heat Transfer

Lecture Note (2) – Conduction (Chapter 2 ~ 5)

Prof. Changmin Son
changminson@vt.edu

Scope of Conduction

Chapter 2 Introduction to Conduction
Derive Heat Diffusion Equation

Chapter 3 1-D Conduction
Simplified problems/solution

Chapter 4 2-D Conduction
***F**inite **E**lement **M**ethod*

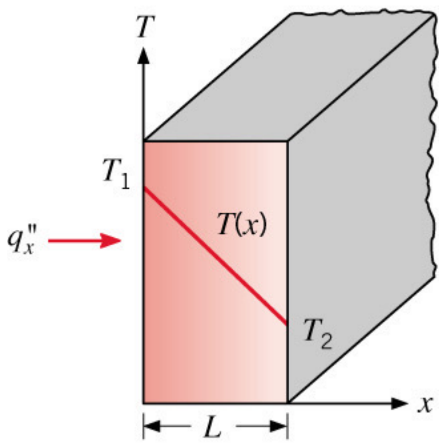
Chapter 5 Transient Conduction
Unsteady
Lumped Capacitance Method
Semi-infinite Solid

Chapter 2

Introduction to Conduction

Conduction

Observation → Concept/Simplification → Theory (equations)
(assumption/modeling)



$$q'' = \frac{q_x}{A_x} = -k_x \times \frac{dT_x}{dx}$$

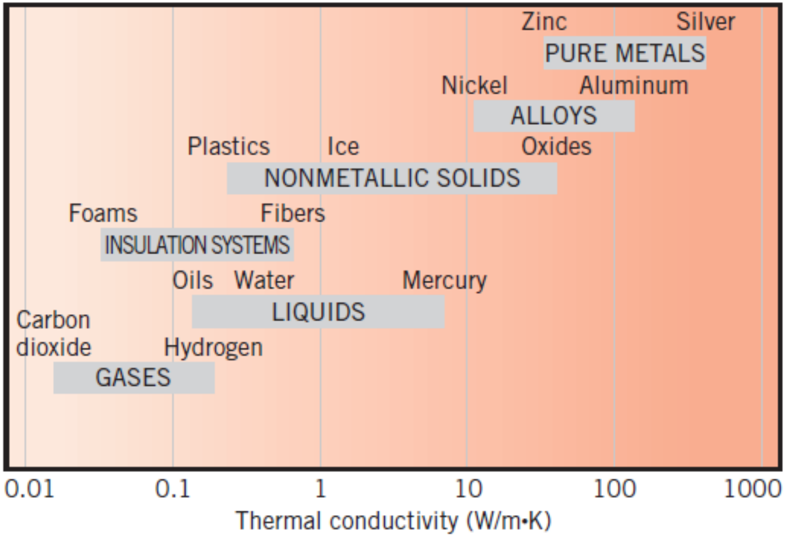
General Form of Conduction Rate Equation:

"Fourier's Law"

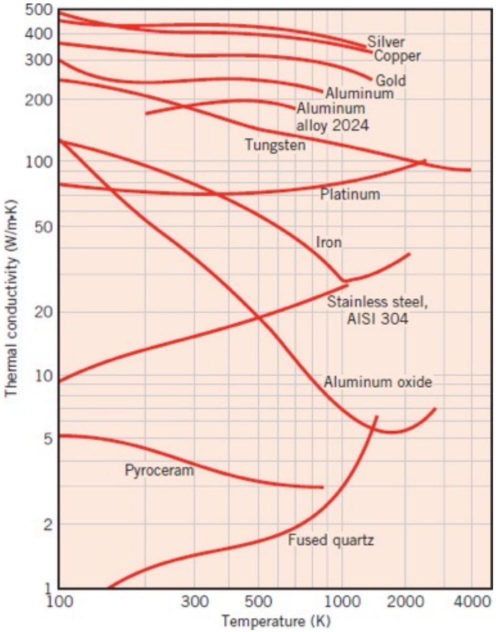
$$q'' = -k\left(\frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} + \frac{\partial T}{\partial z}\vec{k}\right)$$

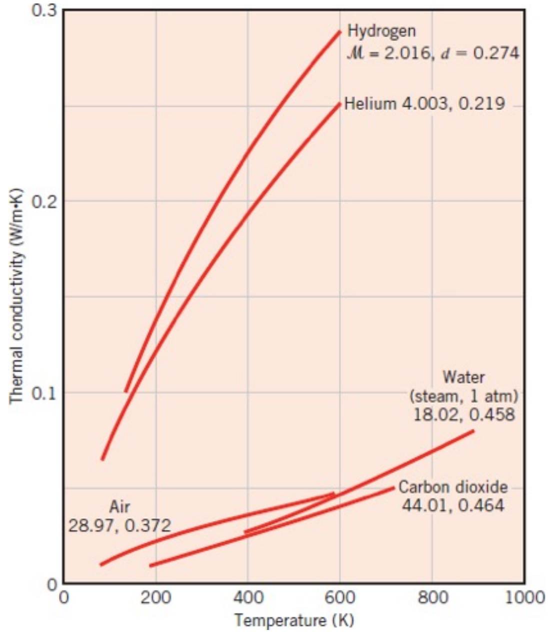
where, $k = k_x = k_y = k_z$

Thermal Conductivity (k) – Material Property

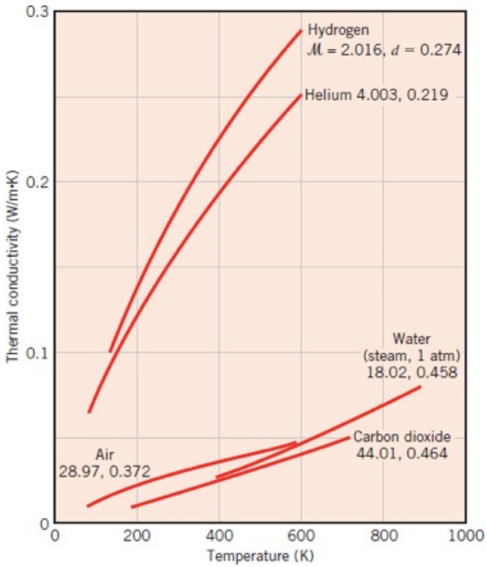


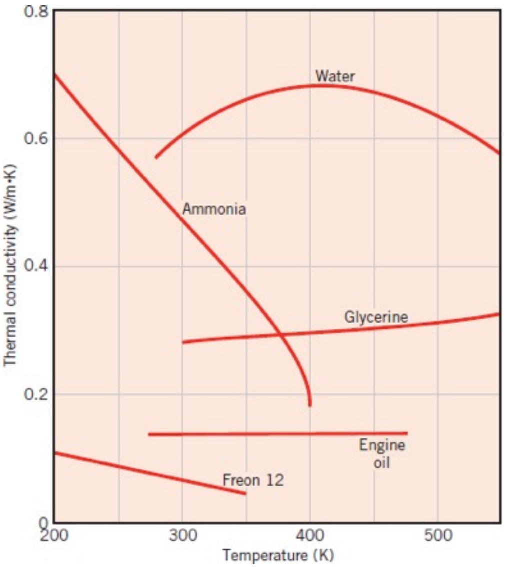
Thermal Conductivity (*k*) – Solid vs. Fluid





Thermal Conductivity (k) – Fluid (Gas vs. Liquid)



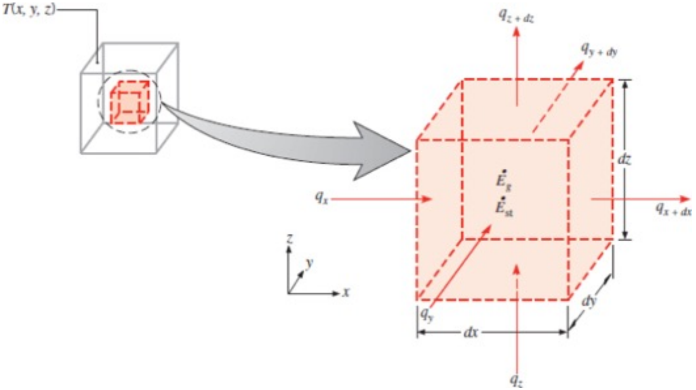


Heat Diffusion Equation

Heat Diffusion Equation – Cartesian Coordinate

Conservation of Energy
(Energy Balance)

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$



$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x}dx = q_x + \frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x}dydz\right)dx$$

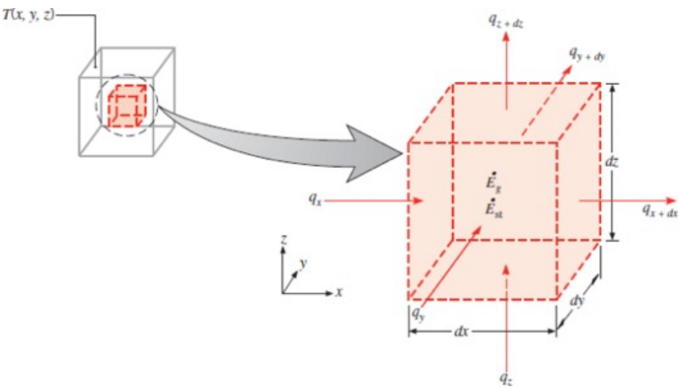
$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y}dy = q_y + \frac{\partial}{\partial y}\left(-k\frac{\partial T}{\partial y}dxdz\right)dy$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z}dz = q_z + \frac{\partial}{\partial z}\left(-k\frac{\partial T}{\partial z}dxdy\right)dz$$

$$\dot{E}_{in} - \dot{E}_{out} = \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right]dxdydz$$

Heat Diffusion Equation – Cartesian Coordinate

Energy Balance $\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$



$$\dot{E}_g = \dot{q} \times dxdydz$$

$$\dot{E}_{st} = \rho c_v \frac{\partial T}{\partial t} \times dxdydz$$

$$= \rho c_p \frac{\partial T}{\partial t} \times dxdydz$$

Heat Diffusion Equation – Cartesian Coordinate

Energy Balance $\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$

$$\dot{E}_{in} - \dot{E}_{out} = \left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] dxdydz$$

$$\dot{E}_g = \dot{q} \times dxdydz$$

$$\dot{E}_{st} = \rho c_v \frac{\partial T}{\partial t} \times dxdydz = \rho c_p \frac{\partial T}{\partial t} \times dxdydz$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

Equation 2.19

Heat Diffusion Equation – Cartesian Coordinate

Energy Balance $\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

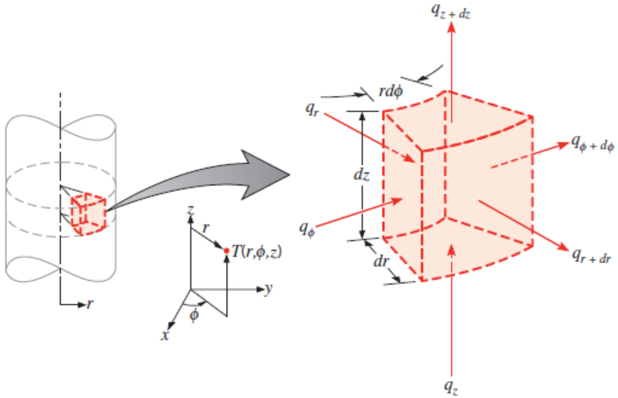
Equation 2.19

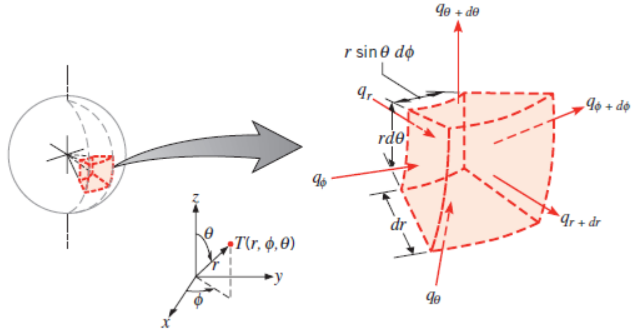
if k = constant, $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Equation 2.21

where, $\alpha = \frac{k}{\rho C_p}$ $[m^2/s]$: thermal diffusivity

Heat Diffusion Equation – Cylindrical/Spherical





Cylindrical Coordinate

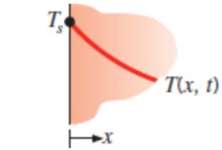
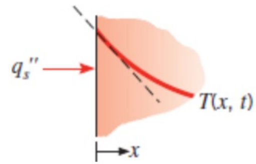
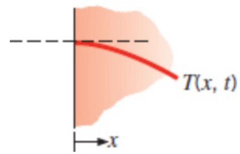
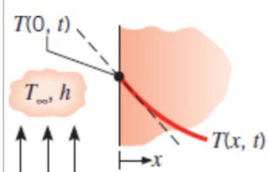
Equation 2.26

Spherical Coordinate

Equation 2.29

Heat Diffusion Equation –Boundary Dependence

TABLE 2.2 Boundary conditions for the heat diffusion equation at the surface ($x = 0$)

1. Constant surface temperature $T(0,t) = T_s$ (2.31)	
2. Constant surface heat flux (a) Finite heat flux $-k \frac{\partial T}{\partial x}\Big\|_{x=0} = q_s''$ (2.32)	
(b) Adiabatic or insulated surface $\frac{\partial T}{\partial x}\Big\|_{x=0} = 0$ (2.33)	
3. Convection surface condition (2.34) $-k \frac{\partial T}{\partial x}\Big\|_{x=0} = h\left[T_\infty - T(0,t)\right]$	

$$q_x''(0) = -k \frac{\partial T}{\partial x}\Big|_{x=0} = q_s''$$

Chapter 3

One-Dimensional Steady-State Conduction

Heat Diffusion Equation – Cartesian Coordinate

Energy Balance $\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

Equation 2.19

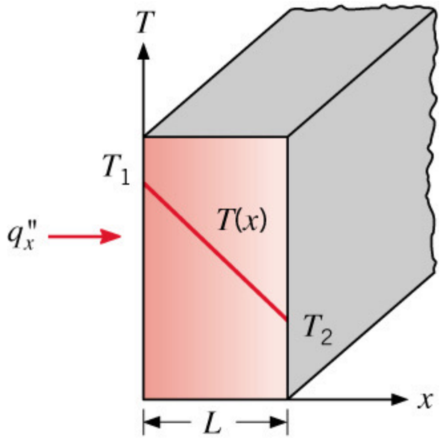
if *k*=constant, $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Equation 2.21

where, $\alpha = \frac{k}{\rho C_p}$ $[m^2/s]$: thermal diffusivity

Conduction with No Heat Generation, 1D & Steady

Q > Is the temperature profile linear?



(1) Heat Diffusion Equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

(2) 1D, steady state, & no heat generation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)=0$$

(3) integrating twice & k = constant

$$T(x)=c_1x+c_2 \qquad \text{(General Solution)}$$

(4) two boundary conditions are

$$\left.\begin{array}{l} T(0)=T_1 \\ T(L)=T_2 \end{array}\right\} \quad c_2=T_1, \quad c_1=\frac{T_2-T_1}{L}$$

(5) therefore,

$$\boxed{T(x)=\frac{T_2-T_1}{L}x+T_1} \qquad \text{(Exact Solution)}$$

Ans > Hence, the temperature distribution is linear function of x.

The Thermal Resistance Concept

Conduction, $$q'' = -k \times \frac{dT}{dx} = k\frac{(T_1 - T_2)}{x_2 - x_1} = k\frac{(T_1 - T_2)}{L}$$

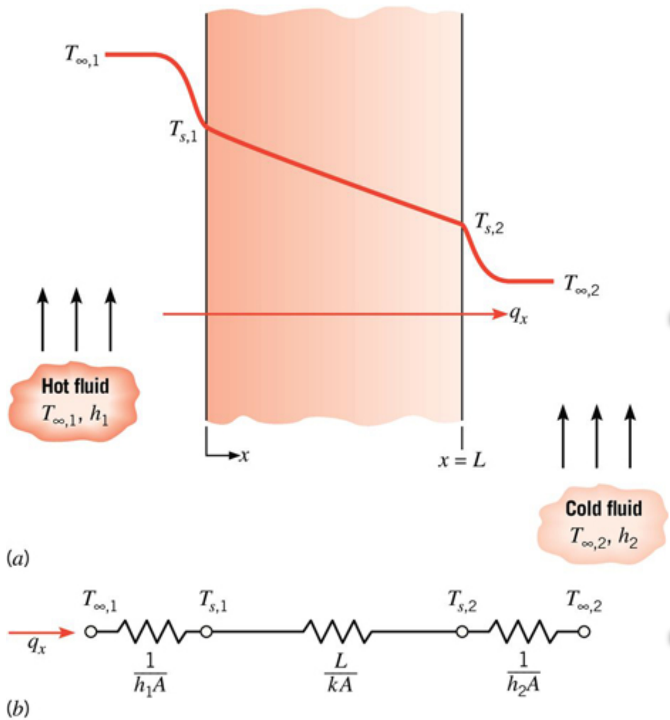
$$q = kA\frac{(T_1 - T_2)}{L} = \frac{\Delta T}{R_t} \quad \Rightarrow \quad R_t = \frac{L}{kA}$$

Convection, $$q'' = h \times \Delta T = h(T_s - T_\infty)$$

$$q = hA(T_s - T_\infty) = \frac{\Delta T}{R_t} \quad \Rightarrow \quad R_t = \frac{1}{hA}$$

Overall heat transfer coefficient, U $\Rightarrow$ $$R_{tot} = \sum R_t = \frac{1}{UA}$$

Thermal Resistance



$$q = q'' \times A = \frac{\Delta T}{R_{tot}}$$

$$R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

FIGURE 3.1 Heat transfer through a plane wall. (*a*) Temperature distribution. (*b*) Equivalent thermal circuit.

Thermal Resistance – Multiple Layers of Walls (1)

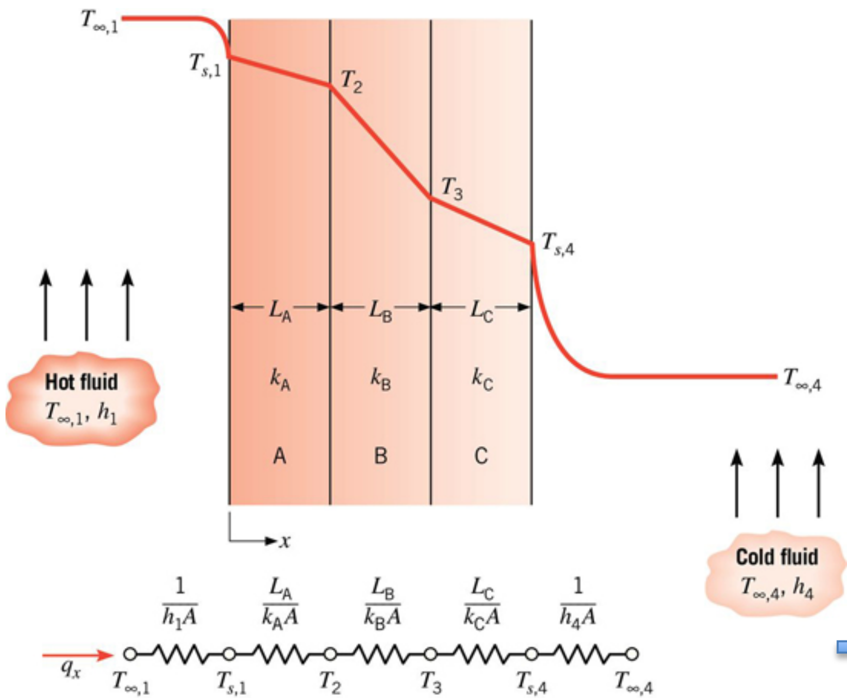


FIGURE 3.2 Equivalent thermal circuit for a series composite wall.

$$q = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t}$$

Thermal Resistance – Multiple Layers of Walls (2)

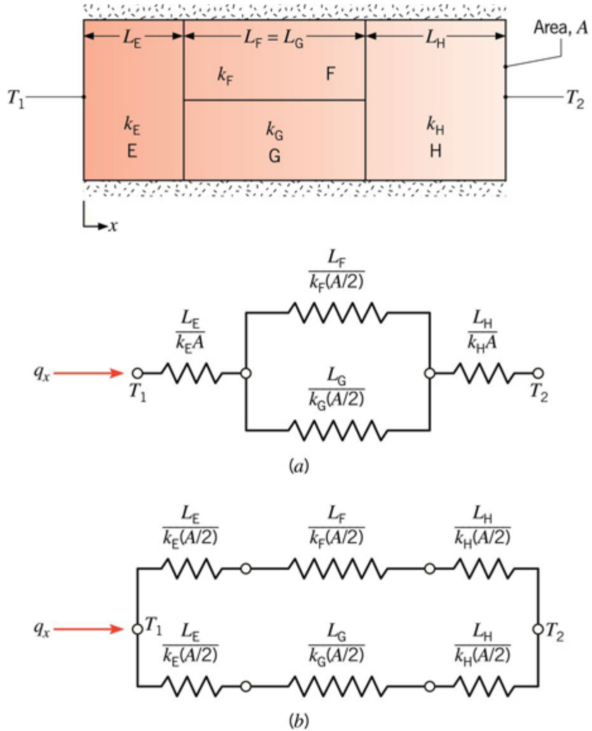


FIGURE 3.3 Equivalent thermal circuits for a series–parallel composite wall.

Contact Resistance – Practical Issues (1)

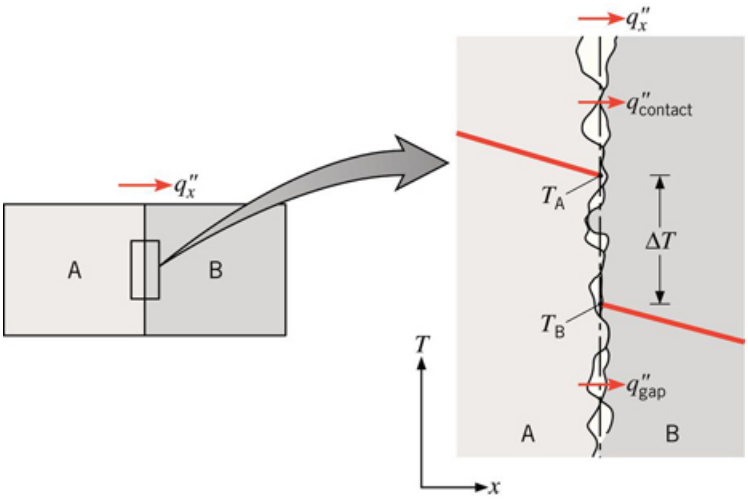


FIGURE 3.4 Temperature drop due to thermal contact resistance.

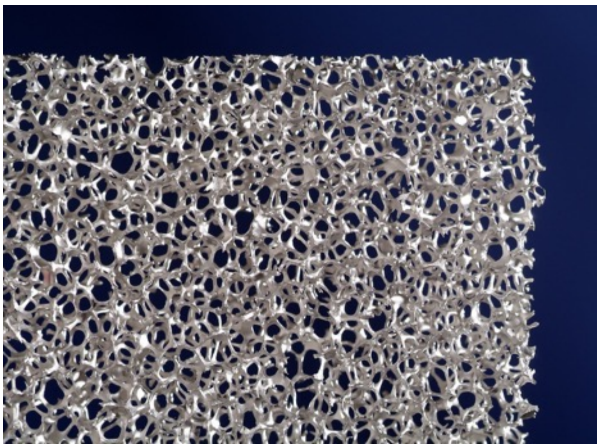
$$R''_{t,c} = \frac{T_A - T_B}{q_x''} \quad (3.20)$$

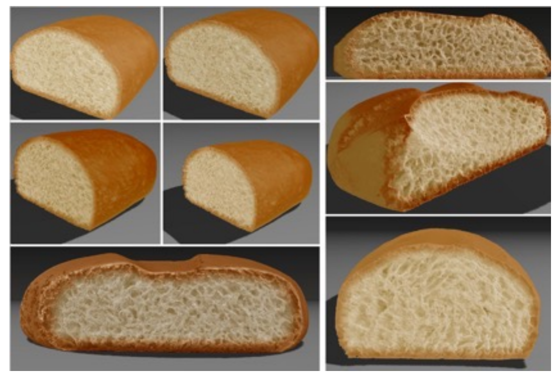
Contact Resistance – Practical Issues (2)

TABLE 3.1 Thermal contact resistance for (*a*) metallic interfaces under vacuum conditions and (*b*) aluminum interface (10-μm surface roughness, 10^5 N/m^2) with different interfacial fluids [1]

Thermal Resistance, $R''_{t,c} \times 10^4 (\mathrm{m^2 \cdot K/W})$				
(*a*) Vacuum Interface			**(*b*) Interfacial Fluid**	
Contact pressure	100 kN/m^2	10,000 kN/m^2	Air	2.75
Stainless steel	6–25	0.7–4.0	Helium	1.05
Copper	1–10	0.1–0.5	Hydrogen	0.720
Magnesium	1.5–3.5	0.2–0.4	Silicone oil	0.525
Aluminum	1.5–5.0	0.2–0.4	Glycerine	0.265

Porous Material – Useful Applications





Porous Material – Thermal Conductivity

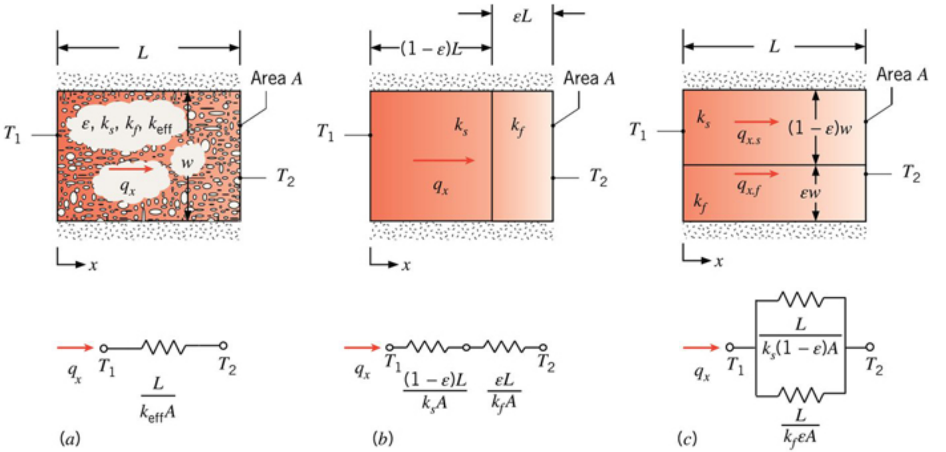


FIGURE 3.5 A porous medium. (*a*) The medium and its properties. (*b*) Series thermal resistance representation. (*c*) Parallel resistance representation.

$$q_x = \frac{k_{\text{eff}}A}{L}(T_1 - T_2) \qquad (3.21)$$

Conduction with Area Change

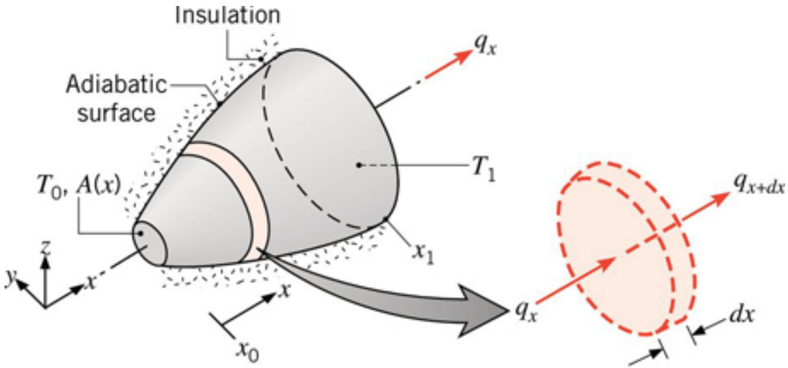
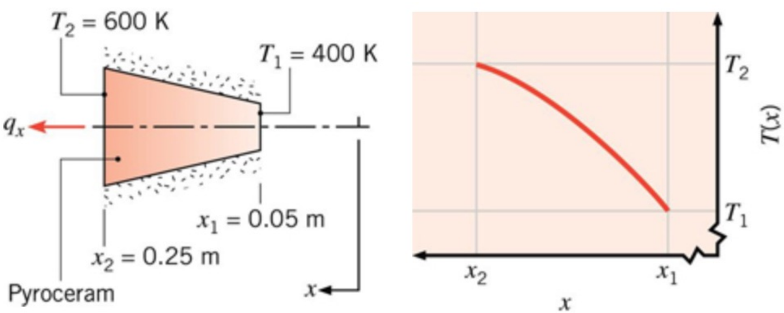


FIGURE 3.6 System with a constant conduction heat transfer rate.

$$q_x = -\int_{x_0}^{x} k(T)A(x)\frac{dT}{dx}$$

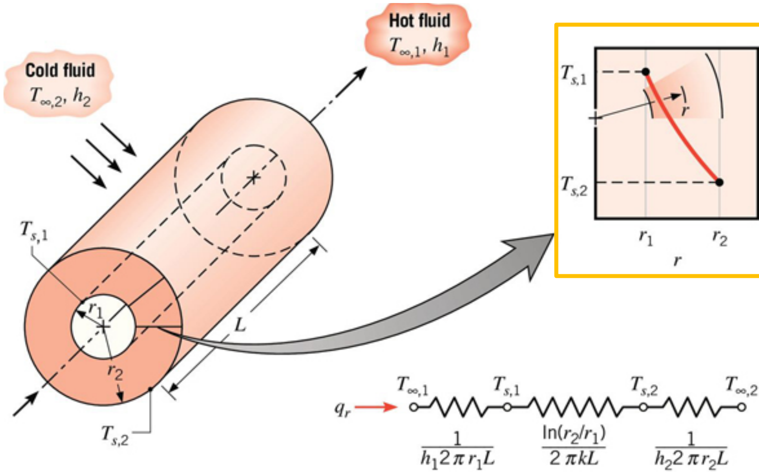
$$q_x \int_{x_0}^{x} \frac{dx}{A(x)} = -\int_{x_0}^{x} k(T)dT$$

Example 3.5



$$T(x) = T_1 + (T_1 - T_2)\left[\frac{(1/x) - (1/x_1)}{(1/x_1) - (1/x_2)}\right]$$

Conduction – Cylindrical Coordinate



$$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_1)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}$$

Equation 3.31

FIGURE 3.7 Hollow cylinder with convective surface conditions.

Boundary Conditions

$$T(r_1) = T_{s,1} \quad \& \quad T(r_1) = T_{s,2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

Equation 2.26

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = 0$$

Equation 3.28

General Solution $T(r) = C_1 \ln r + C_2$

Equation 3.30

Cylindrical Coordinate – Temperature Distribution

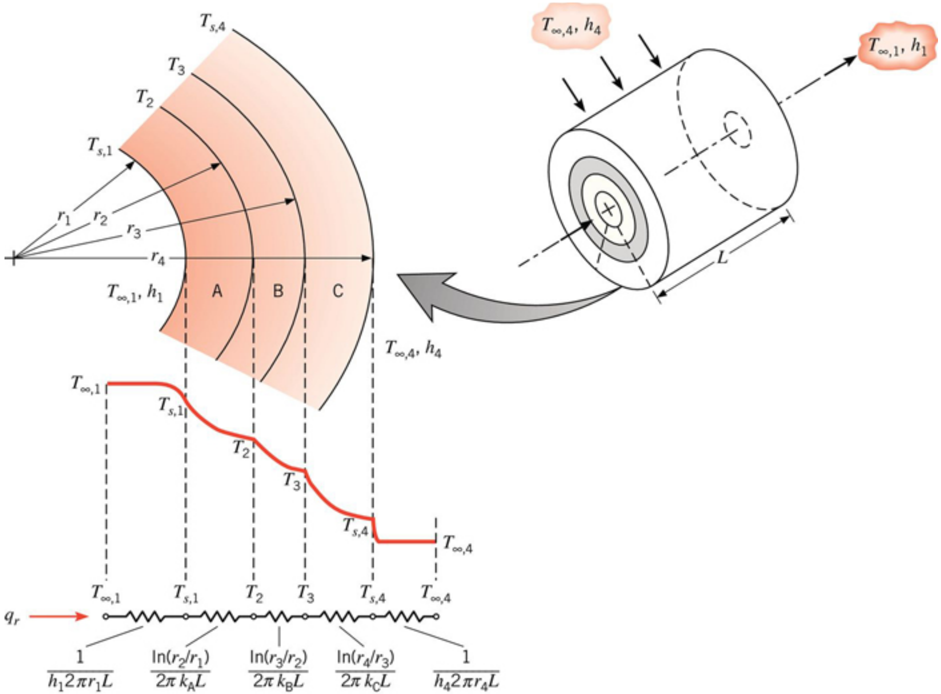


FIGURE 3.8 Temperature distribution for a composite cylindrical wall.

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{2\pi r_1 L h_1} + \frac{\ln\,(r_2/r_1)}{2\pi k_A L} + \frac{\ln\,(r_3/r_2)}{2\pi k_B L} + \frac{\ln\,(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}} \quad (3.34)$$

Example 3.6

Conduction – Spherical Coordinate

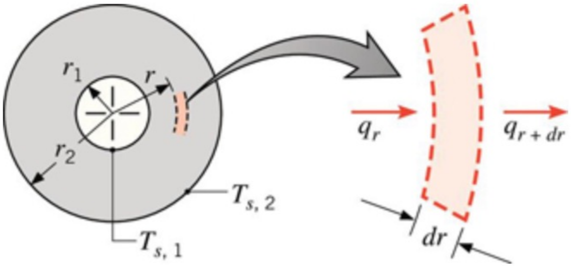


FIGURE 3.9 Conduction in a spherical shell.

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$

$$\frac{q_x}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{T_{s,1}}^{T_{s,2}} k(T)dT$$

if k = constant

$$q_r = \frac{4\pi k\left(T_{s,1} - T_{s,2}\right)}{(1/r_1) - (1/r_2)}$$

$$R_{t,cond} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Summary of 1D Conduction Results

TABLE 3.3 One-dimensional, steady-state solutions to the heat equation with no generation

	Plane Wall	**Cylindrical Wall**[a]	**Spherical Wall**[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\text{ cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

Conduction with Heat Generation

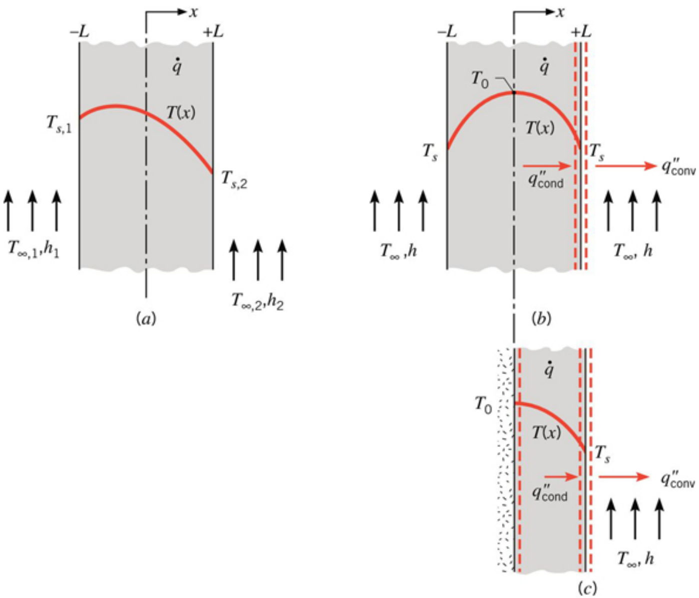


FIGURE 3.10 Conduction in a plane wall with uniform heat generation. (*a*) Asymmetrical boundary conditions. (*b*) Symmetrical boundary conditions. (*c*) Adiabatic surface at midplane.

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = 0$$

$$\frac{\partial^2 T}{\partial^2 x} + \frac{\dot{q}}{k} = 0 \quad \text{if } k = \text{constant}$$

$$T(x) = \frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$

Equation 3.45

Boundary Conditions

$$T(-L) = T_{s,1} \quad \& \quad T(L) = T_{s,2}$$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2}$$

Equation 3.46

Example 3.7

Conduction with Heat Generation – Cylindrical Coordinate

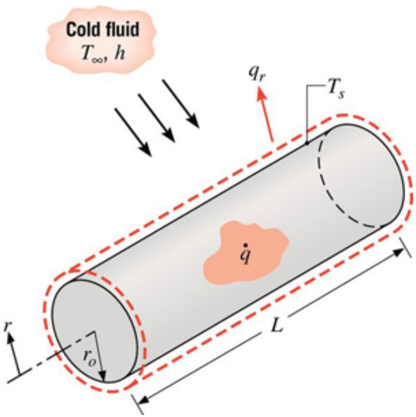


FIGURE 3.11 Conduction in a solid cylinder with uniform heat generation.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\dot{q}=0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{q}}{k}=0 \quad \text{if } k = \text{constant}$$

$$T(r)=-\frac{\dot{q}}{4k}r^2+C_1\ln r+C_2$$

Equation 3.56

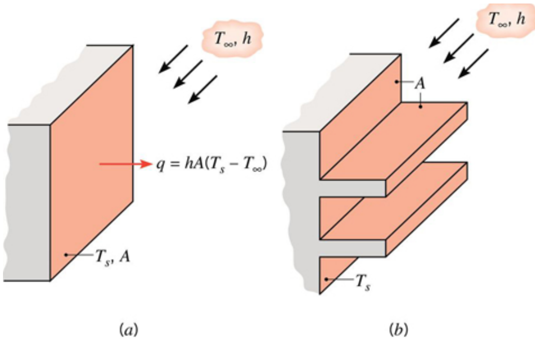
Boundary Conditions

$$\left.\frac{dT}{dr}\right|_{r=0}=0 \quad \& \quad T(r_o)=T_s$$

$$T(x)=\frac{\dot{q}r_0^2}{4k}\left(1-\frac{r^2}{r_0^2}\right)+T_s$$

Equation 3.58

Extended Surface Approach



Use of fins to enhance heat transfer from a plane wall. (*a*) Bare surface. (*b*) Finned surface.

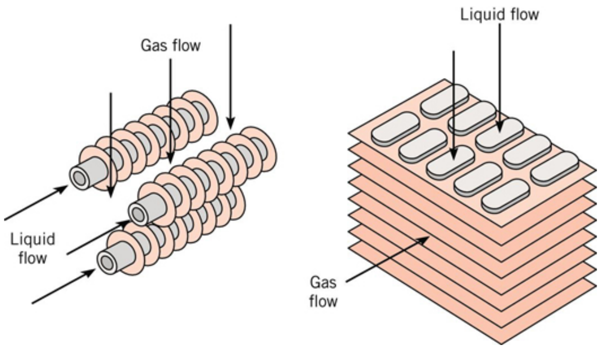


FIGURE 3.14 Schematic of typical finned-tube heat exchangers.

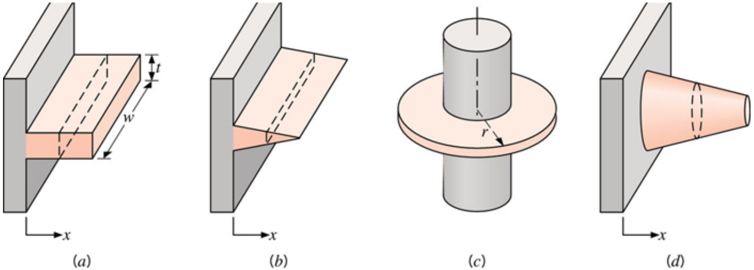


FIGURE 3.15 Fin configurations. (*a*) Straight fin of uniform cross section. (*b*) Straight fin of nonuniform cross section. (*c*) Annular fin. (*d*) Pin fin.

Extended Surface (fin) Model

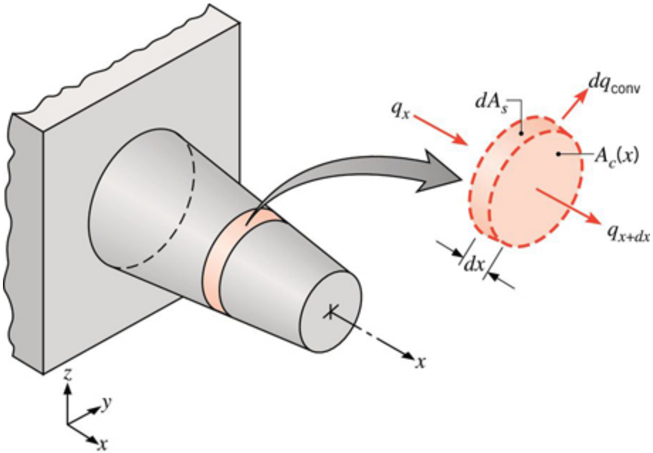


FIGURE 3.16 Energy balance for an extended surface.

$$q_x = q_{x+dx} + dq_{conv}$$ Energy Balance

Conduction

$$q_x = -kA_c \frac{dT}{dx}$$

$$q_{x+dx} = q_x + \frac{dq_x}{dx} dx$$

$$= -kA_c \frac{dT}{dx} + \frac{d}{dx}\left(-kA_c \frac{dT}{dx}\right) dx$$

Convection

$$dq_{conv} = h dA_s (T - T_\infty)$$
$$= hPdx\,(T - T_\infty)$$

$$\frac{d}{dx}\left(A_c \frac{dT}{dx}\right) - \frac{h}{k} P (T - T_\infty) = 0$$

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{1}{A_c}\frac{dA_c}{dx}\frac{dT}{dx} - \frac{hP}{kA_c}(T - T_\infty) = 0}$$

Equation 3.66

Fins with uniform cross sectional area (single fin)

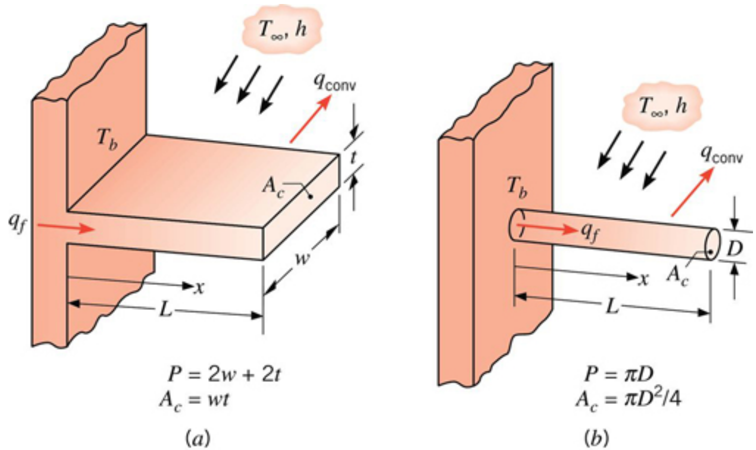


FIGURE 3.17 Straight fins of uniform cross section. (*a*) Rectangular fin. (*b*) Pin fin.

$$\frac{\partial^2 T}{\partial^2 x} - \frac{hP}{kA_c}(T - T_\infty) = 0 \qquad \text{defining} \quad \theta(x) \equiv T(x) - T_\infty$$ Equation 3.67

$$\frac{\partial^2 \theta}{\partial^2 x} - m\theta = 0 \qquad \text{where,} \ m^2 = \frac{hP}{kA_c}$$ Equation 3.67

General Solution

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$ Equation 3.71

$$= C_1 \exp(mx) + C_2 \exp(-mx)$$

Fins with uniform cross sectional area

(four different cases at $x = L$)

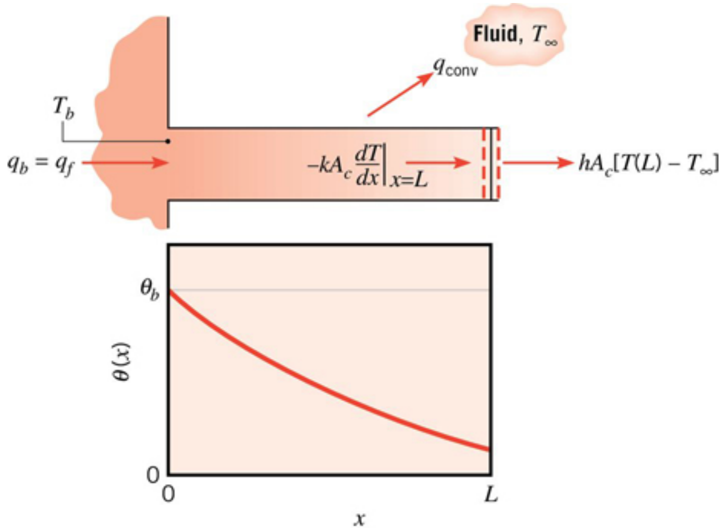


FIGURE 3.18 Conduction and convection in a fin of uniform cross section.

Example 3.9

Fins with uniform cross sectional area

TABLE 3.4 **Temperature distribution and heat rates for fins of uniform cross section**

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b		Fin Heat Transfer Rate q_f	
A	Convection: $h\theta(L) = -kd\theta/dx\vert_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$	(3.75)	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$	(3.77)
B	Adiabatic: $d\theta/dx\vert_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	(3.80)	$M \tanh mL$	(3.81)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$	(3.82)	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$	(3.83)
D	Infinite fin ($L \to \infty$): $\theta(L) = 0$	e^{-mx}	(3.84)	M	(3.85)
$\theta \equiv T - T_\infty$ $\theta_b = \theta(0) = T_b - T_\infty$	$m^2 \equiv hP/kA_c$ $M \equiv \sqrt{hPkA_c}\theta_b$	A table of hyperbolic functions is given in Appendix B.1.			

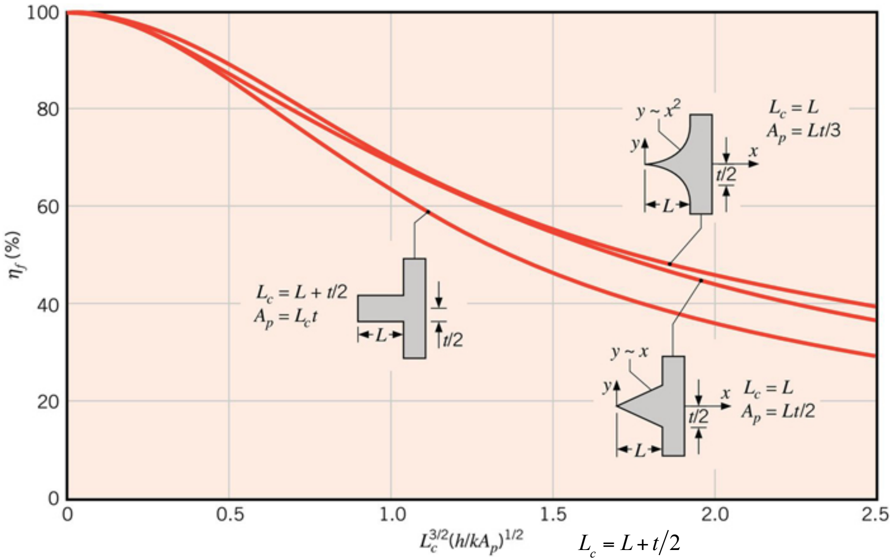
Fin Performance

Effectiveness $$\varepsilon_f = \frac{q_f}{hA_{c,b}\theta_b}$$

Resistance $$R_{t,f} = \frac{\theta_b}{q_f}$$

Efficiency $$\eta_f = \frac{q_f}{q_{max}} = \frac{q_f}{hA_f(T_b - T_\infty)} = \frac{q_f}{hA_f\theta_b}$$

Fin Performance – Corrected Length and Efficiency



$L_c = L + t/2$

FIGURE 3.19 Efficiency of straight fins (rectangular, triangular, and parabolic profiles).

Fin Performance – Corrected Length and Efficiency

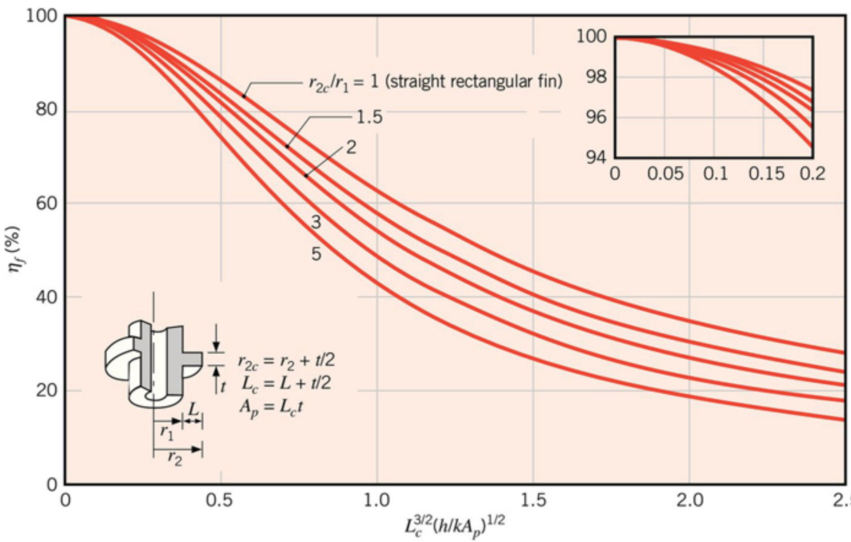


FIGURE 3.20 Efficiency of annular fins of rectangular profile.

Fins – Nonuniform Cross-Sectional Area

Special case of Figure 3.20

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{A_c}\frac{dA_c}{dx}\frac{dT}{dx} - \frac{hP}{kA_c}(T - T_\infty) = 0$$ Equation 3.66

assume $A_c = 2\pi rt,\ A_s = 2\pi(r^2 - r_1^2)$ and replacing x by r

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kt}(T - T_\infty) = 0$$

with, $m^2 = 2h/kt\,,\ \ \theta \equiv T - T_\infty$

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{d\theta}{dr} - m^2\theta = 0$$

solution has the expression of modified Bessel equation

$$\theta(r) = C_1 I_0(mr) + C_2 K_0(mr)$$

Table 3.5 Efficiency of common fin shapes

Fin arrays – Overall Surface Efficiency

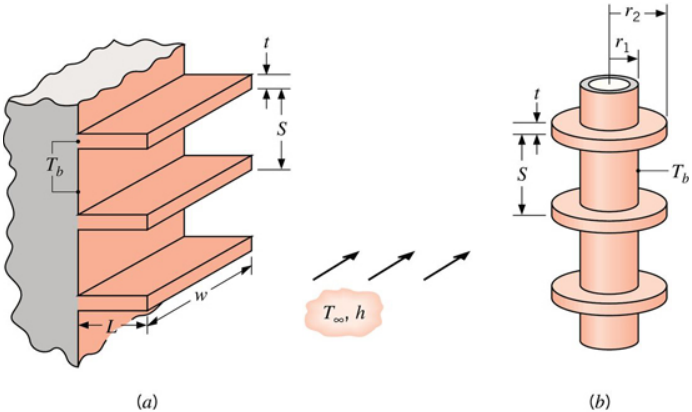


FIGURE 3.21 Representative fin arrays. (*a*) Rectangular fins. (*b*) Annular fins.

$$\eta_o = \frac{q_t}{q_{max}} = \frac{q_t}{hA_t\theta_b}$$

$$R_{t,o} = \frac{\theta_b}{q_t} = \frac{1}{\eta_o h A_t}$$

Fin arrays – Thermal Resistance Model

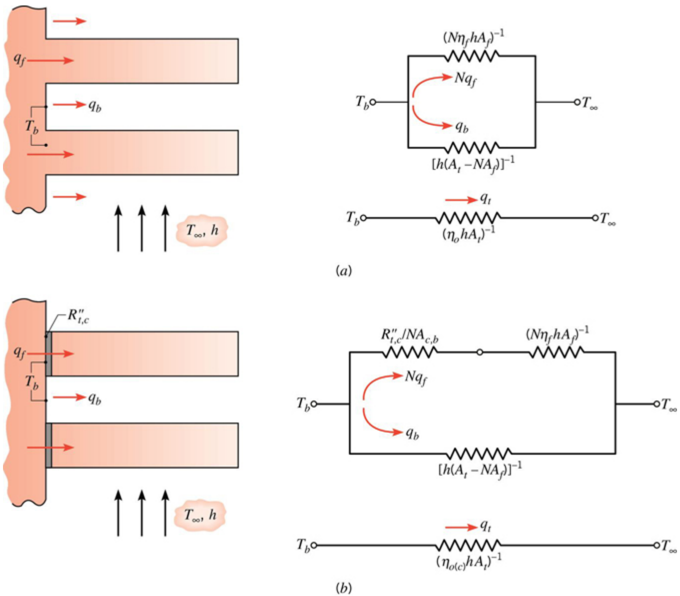


FIGURE 3.22 Fin array and thermal circuit. (*a*) Fins that are integral with the base. (*b*) Fins that are attached to the base.

Example 3.10

Others

- Boiheat
- Thermoelectric Power Generation

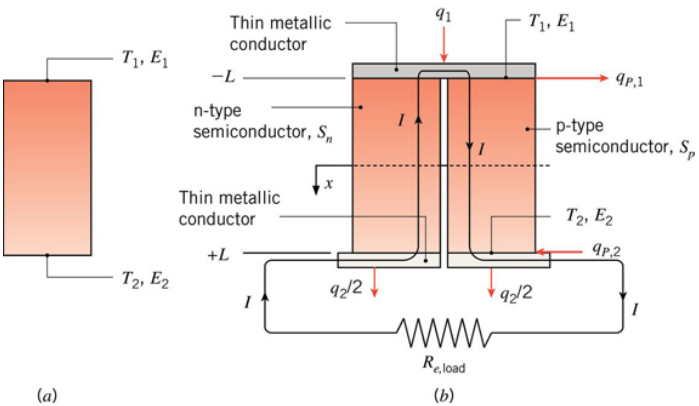
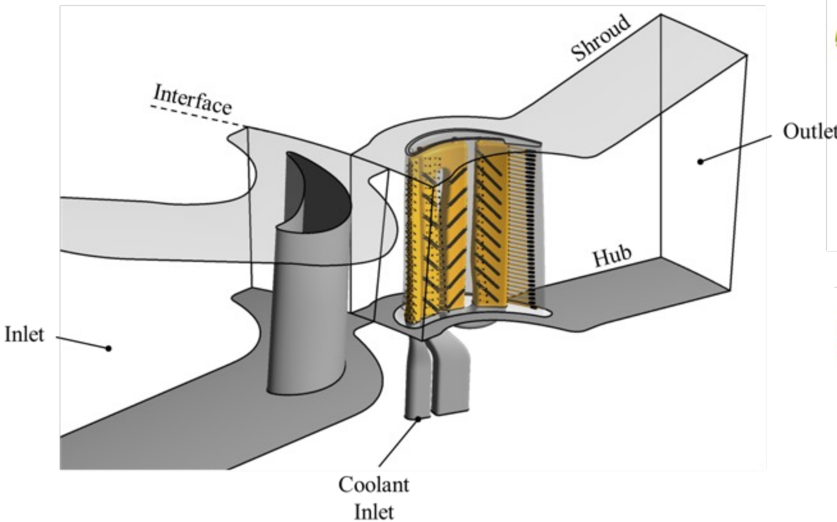


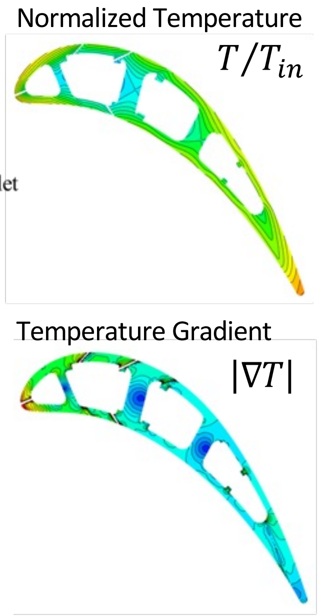
FIGURE 3.23 Thermoelectric phenomena. (*a*) The Seebeck effect. (*b*) A simplified thermoelectric circuit consisting of one pair ($N = 1$) of semiconducting pellets.

Chapter 4

Two-Dimensional Steady-State Conduction

A real problem and FEM analysis





Heat Diffusion Equation – Cartesian Coordinate

Energy Balance $\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

Equation 2.19

if *k*=constant, $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Equation 2.21

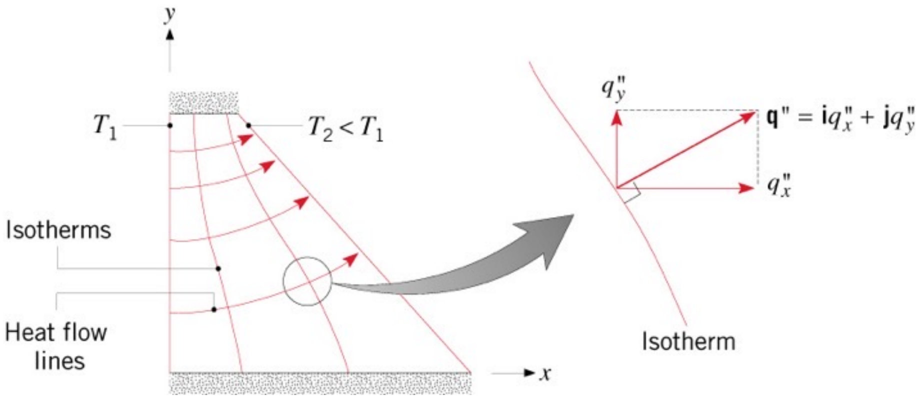
where, $\alpha = \frac{k}{\rho C_p}$ $[m^2/s]$: thermal diffusivity

Two-dimensional conduction

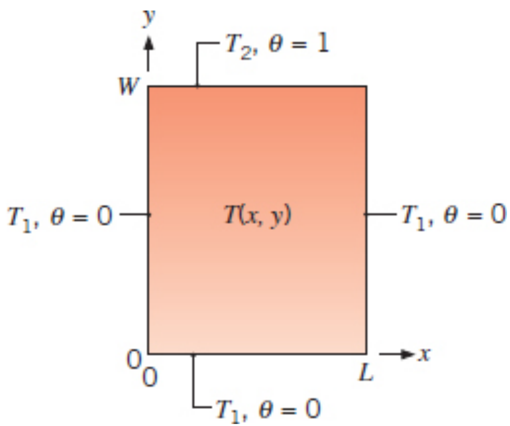
Steady-state with no heat generation and constant thermal condcutivity

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Heat transfer in a long, prismatic solid with two isothermal surfaces and two insulated surfaces:



Two-dimensional conduction



Boundary Conditions
$\theta(0, y) = 0$, and $\theta(x, 0) = 0$
$\theta(L, y) = 0$, and $\theta(x, W) = 1$

FIGURE 4.2 Two-dimensional conduction in a thin rectangular plate or a long rectangular rod.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad \Rightarrow \quad \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \qquad where, \theta = \frac{T - T_1}{T_2 - T_1}$$

Two-dimensional conduction – Temp. distribution

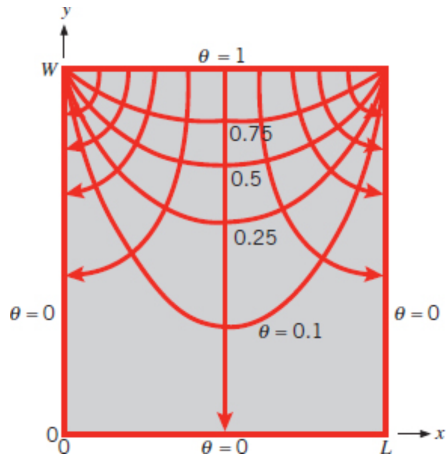


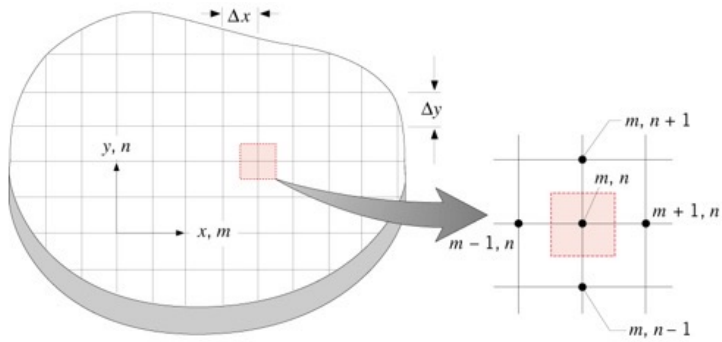
FIGURE 4.3 Isotherms and heat flow lines for two-dimensional conduction in a rectangular plate.

$$\theta(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}\sin\frac{n\pi x}{L}\frac{\sinh(n\pi y/L)}{\sinh(n\pi W/L)}$$ Equation 4.19

$$where, \theta = \frac{T-T_1}{T_2-T_1}$$

The Nodal Network and Finite-Difference Approximation (1)

- The nodal network identifies discrete points at which the temperature is to be determined and uses an *m,n* notation to designate their location.

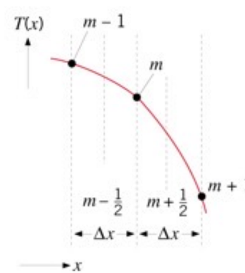


(a)

- A finite-difference approximation is used gradients in the domain.

$$\left.\frac{\partial T}{\partial x}\right|_{m-1/2,\,n} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m+1/2,\,n} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$



(b)

The Nodal Network and Finite-Difference Approximation (2)

Equation 4.29

Derivation of the Finite-Difference Equations (1)

- As a convenience that obviates the need to predetermine the direction of heat flow, assume all heat flows are into the nodal region of interest, and express all heat rates accordingly.

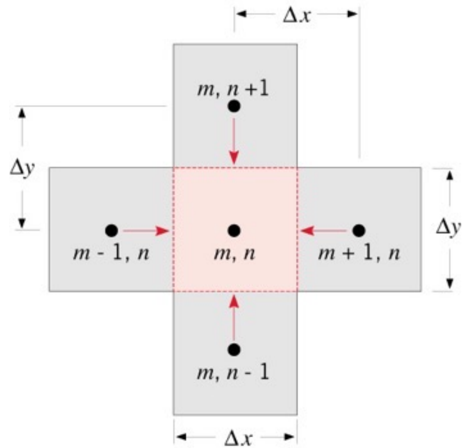
Hence, the **energy balance becomes**: $\dot{E}_{in} + \dot{E}_g = 0$ (4.30)

- Consider application to an *interior nodal point* (one that exchanges heat by conduction with four, equidistant nodal points):

$$\sum_{i=1}^{4} q_{i \to m,n} + \dot{q}(\Delta x \cdot \Delta y \cdot l) = 0$$

where, for example,

$$q_{m-1,n \to m,n} = k(\Delta y \cdot l)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} \qquad (4.31)$$



which leads to

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} + \frac{\dot{q}(\Delta x)^2}{k} - 4T_{m,n} = 0$$

Equation 4.35

Derivation of the Finite-Difference Equations (2)

Figure 4.6

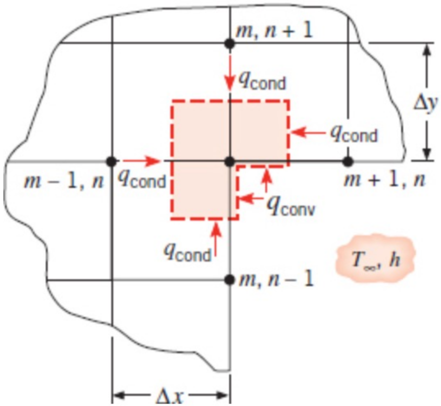


Table 4.2

Example 4.2

Chapter 5

Transient Conduction

Lumped Capacitance Method, Semi-Infinite Solid

Heat Diffusion Equation – Cartesian Coordinate

Energy Balance $\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$

Equation 2.19

if k=constant, $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Equation 2.21

where, $\alpha = \frac{k}{\rho C_p}$ $[m^2/s]$: thermal diffusivity

The Lumped Capacitance Method - Model

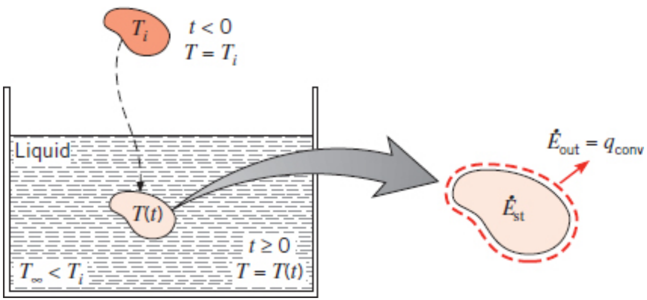


FIGURE 5.1 Cooling of a hot metal forging.

no heat in

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$

Energy Balance

$$-\dot{E}_{out} = \dot{E}_{st}$$

$$-hA(T - T_\infty) = \rho V c_p \frac{dT}{dt}$$

defining $\theta(x) \equiv T(x) - T_\infty$

$$-\theta = \frac{\rho V c_p}{hA_s} \frac{d\theta}{dt}$$

Separating variables

$$-dt = \frac{\rho V c_p}{hA_s} \frac{d\theta}{\theta}$$

The Lumped Capacitance Method - Solution

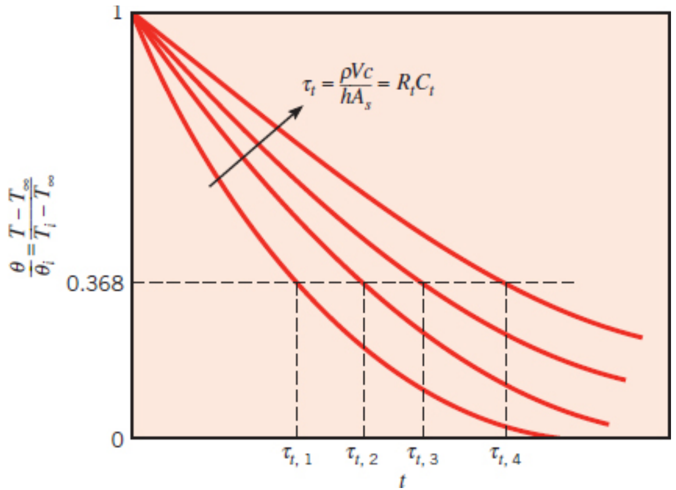


FIGURE 5.2 Transient temperature response of lumped capacitance solids for different thermal time constants τ_i.

constant

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho V c_p}\right)t\right] = \exp\left(-\frac{t}{\tau_t}\right) \qquad \text{Equation 5.6}$$

where, thermal time constant, $\tau_t = \frac{\rho V c_p}{h A_s} = R_t C_t$

and, C_t is lumped thermal capacitance

Validity of the Lumped Capacitance Method – Biot Number

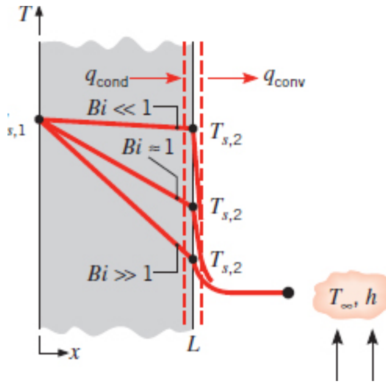


FIGURE 5.3 Effect of Biot number on steady-state temperature distribution in a plane wall with surface convection.

$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)$$

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{(L/kA)}{(1/hA)} = \frac{R_{t,\text{cond}}}{R_{t,\text{conv}}} = \frac{hL}{k} \equiv Bi$$

Equation 5.6

Validity of the Lumped Capacitance Method – Biot Number

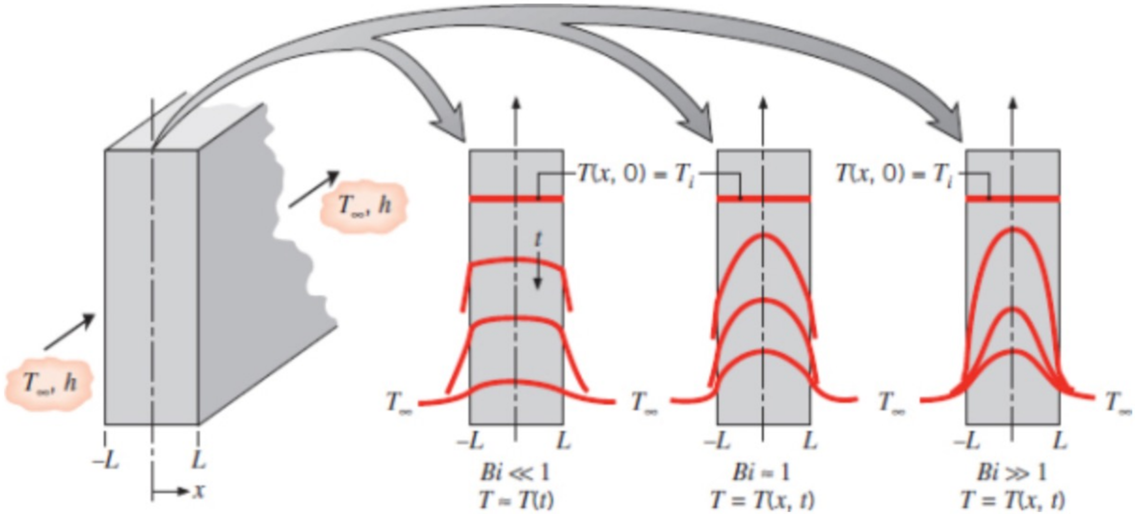
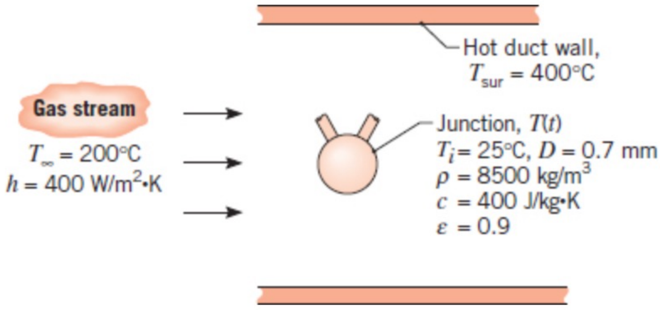


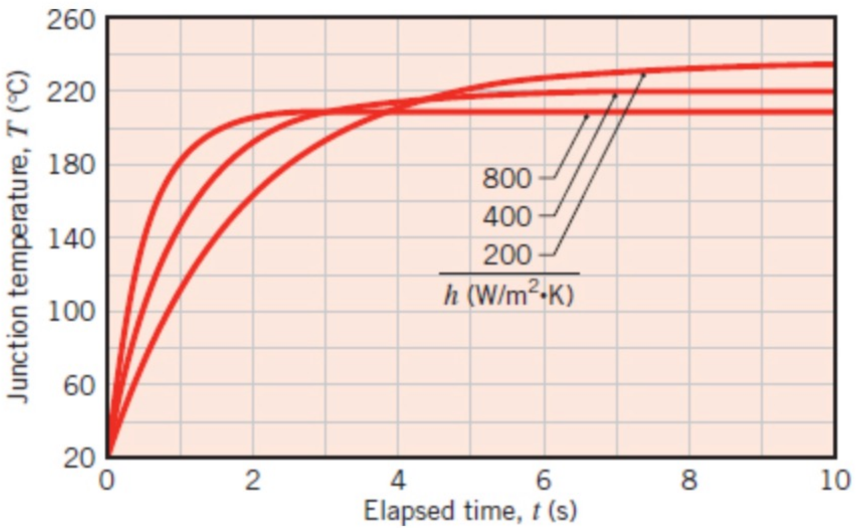
FIGURE 5.4 Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection.

$$Bi = \frac{hL_c}{k} < 0.1$$

Example 5.1

Example 5.2


Hot duct wall,
T_{sur} = 400°C
Gas stream
T_∞ = 200°C
h = 400 W/m²·K
Junction, $T(t)$
T_i = 25°C, D = 0.7 mm
ρ = 8500 kg/m³
c = 400 J/kg·K
ε = 0.9


260
220
180
140
100
60
20
0
2
4
6
8
10
Junction temperature, T (°C)
Elapsed time, t (s)
800
400
200
h (W/m²·K)

The Lumped Capacitance Method - Another form of the Solution

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho V c}\right)t\right] \Longleftrightarrow \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \exp(-Bi \cdot Fo)$$

$$\frac{hA_s t}{\rho V c} = \frac{ht}{\rho c L_c} = \frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L_c^2} = \frac{hL_c}{k}\frac{\alpha t}{L_c^2}$$

$$\frac{hA_s t}{\rho V c} = Bi \cdot Fo$$

} Equation 5.11

where, Fourier Number, (~diminsionaless time) $Fo = \frac{\alpha t}{L_c^2}$

Characteristic Length $\left(\frac{\text{Volume}}{\text{Surface Area}}\right)$, $L_c = \frac{V}{A_s}$

General Lump Capacitance Method - Model

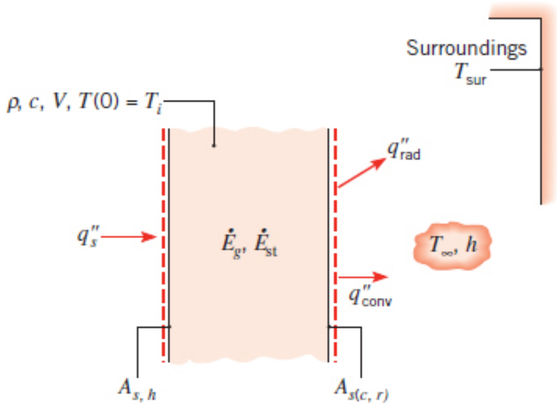


FIGURE 5.5 Control surface for general lumped capacitance analysis.

Energy Balance Equation

$$q''_s A_{s,h} + \dot{E}_g - (q''_{conv} + q''_{rad})A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

$$q''_s A_{s,h} + \dot{E}_g - [h(T - T_\infty) + \varepsilon\sigma(T^4 - T^4_{sur})]A_{s(c,r)} = \rho V c \frac{dT}{dt}$$

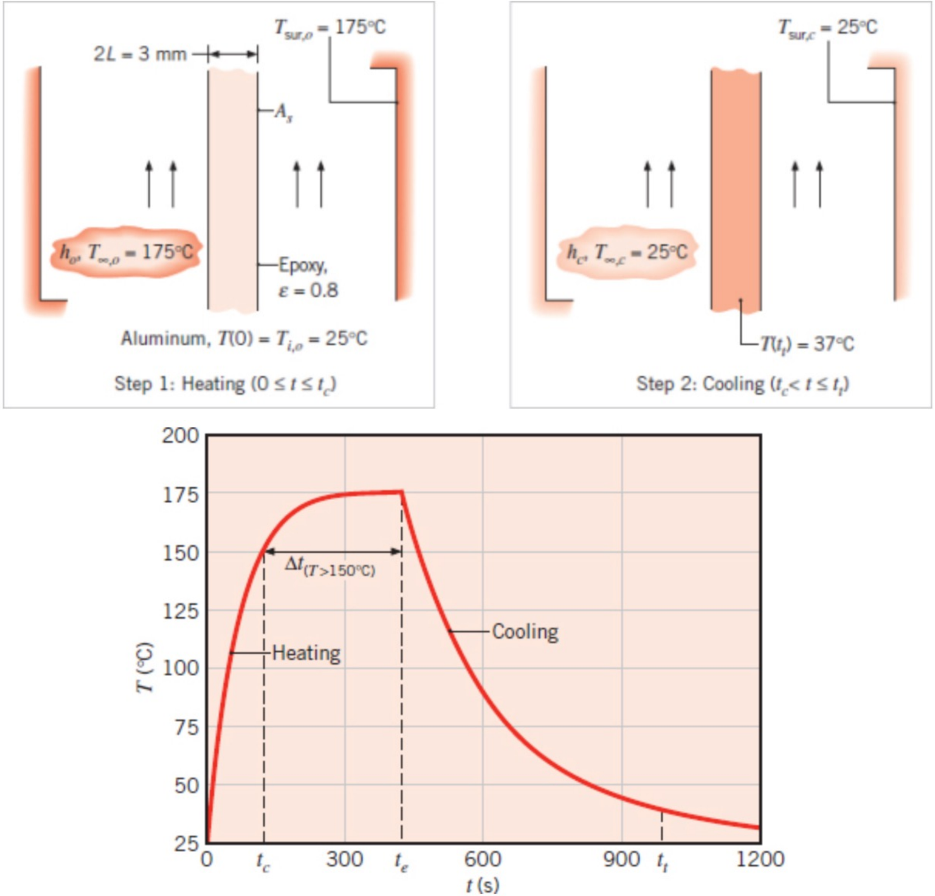
Equation 5.15

General Lump Capacitance Method

Section	Solution	Equations
5.3.1	Radiation only	Equation 5.18
5.3.2	Negligible Radiation	Equation 5.25
5.3.3	Convection only with Variable Convection Coefficient	Equation 5.28
5.3.4	Additional Considerations	-

Example 5.3 & 5.4

Example 5.3



$T_{sur,o}$ = 175°C
2L = 3 mm
A_s
h_o, $T_{\infty,o}$ = 175°C
Epoxy,
ε = 0.8
Aluminum, T(0) = $T_{i,o}$ = 25°C
Step 1: Heating (0 ≤ t ≤ t_c)
$T_{sur,c}$ = 25°C
h_c, $T_{\infty,c}$ = 25°C
$T(t_t)$ = 37°C
Step 2: Cooling (t_c < t ≤ t_t)
200
175
150
125
100
75
50
25
T (°C)
$\Delta t_{(T>150°C)}$
Heating
Cooling
0
t_c
300
t_e
600
900
t_t
1200
t (s)

Spatial Effect

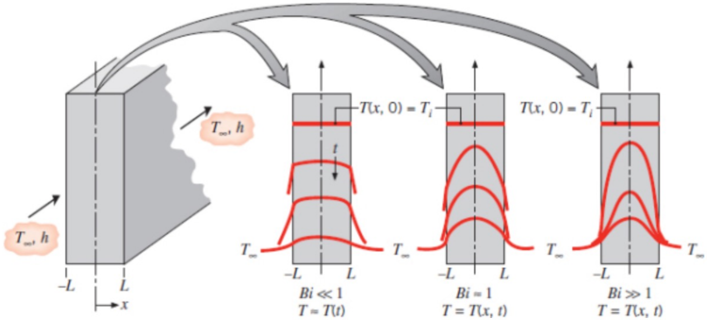


FIGURE 5.4 Transient temperature distributions for different Biot numbers in a plane wall symmetrically cooled by convection.

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ Equation 5.29

$$\frac{\partial^2 \theta^*}{\partial x^{*2}} = \frac{\partial \theta^*}{\partial Fo}$$ Equation 5.37

$$\theta^* = f(x^*, Fo, Bi)$$ Equation 5.41

Plane Wall with Convection

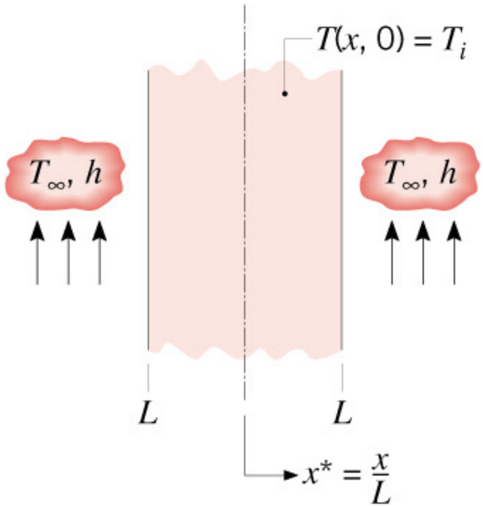


Figure 5.6

$$\theta^* = \sum_{n=1}^{\infty} C_n exp(-\zeta_n^2 Fo) \cos(\zeta_n x^*) \quad \text{Equation 5.42}$$

where, $Fo = \dfrac{\alpha t}{L_c^2}$ and, $C_n = \dfrac{4 \sin \zeta_n}{2\zeta_n + \sin 2\zeta_n}$

Radial Systems with Convection

Infinite Cylinder or Sphere of Radius, r_0

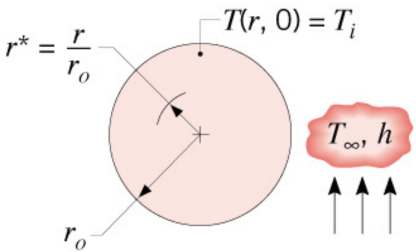


Figure 5.6

$$\theta^* = \sum_{n=1}^{\infty} C_n exp(-\zeta_n^2 Fo) \cos(\zeta_n r^*)$$ Equation 5.50

where, $Fo = \frac{\alpha t}{r_0^2}$ and, $C_n = \frac{2}{\zeta_n} \frac{J_1(\zeta_n)}{J_0^2(\zeta_n) + J_1^2(\zeta_n)}$

J_1 and J_2: Bessel function of first kind

Example 5.5 & 5.6

Semi-Infinite Solid (assumption) - Model

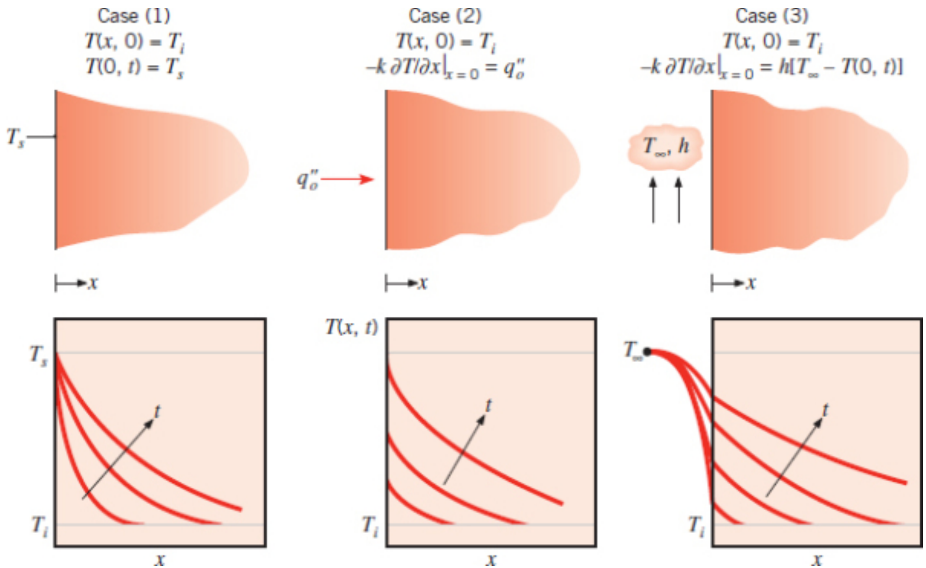


FIGURE 5.7 Transient temperature distributions in a semi-infinite solid for three surface conditions: constant surface temperature, constant surface heat flux, and surface convection.

Semi-Infinite Solid – Derivation (Section 5.7)

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ Equation 5.29

$$\frac{\partial^2 T}{\partial \eta^2} = -2\eta\frac{\partial T}{\partial \eta}$$ Equation 5.57

where, similarity variable, $\eta \equiv \frac{x}{\sqrt{4\alpha t}}$

Semi-Infinite Solid - Solution

Case 1 Constant Surface Temperature: $T(0, t) = T_s$

$$\frac{T(x,t)-T_s}{T_i-T_s} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{5.60}$$

$$q_s''(t) = \frac{k(T_s-T_i)}{\sqrt{\pi\alpha t}} \tag{5.61}$$

Case 2 Constant Surface Heat Flux: $q_s'' = q_o''$

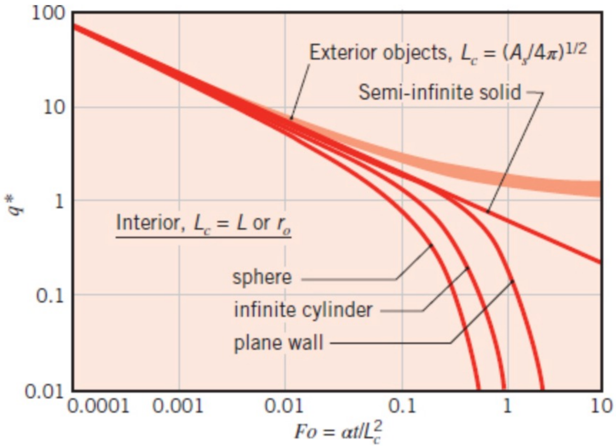
$$T(x,t)-T_i = \frac{2q_o''(\alpha t/\pi)^{1/2}}{k}\exp\left(\frac{-x^2}{4\alpha t}\right) - \frac{q_o'' x}{k}\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \tag{5.62}$$

Case 3 Surface Convection: $-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h[T_\infty - T(0,t)]$

$$\frac{T(x,t)-T_i}{T_\infty-T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \left[\exp\left(\frac{hx}{k}+\frac{h^2\alpha t}{k^2}\right)\right]\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}+\frac{h\sqrt{\alpha t}}{k}\right)\right] \tag{5.63}$$

Case 1 & 2

Constant Surface Temperature



Constant Surface Heat Flux

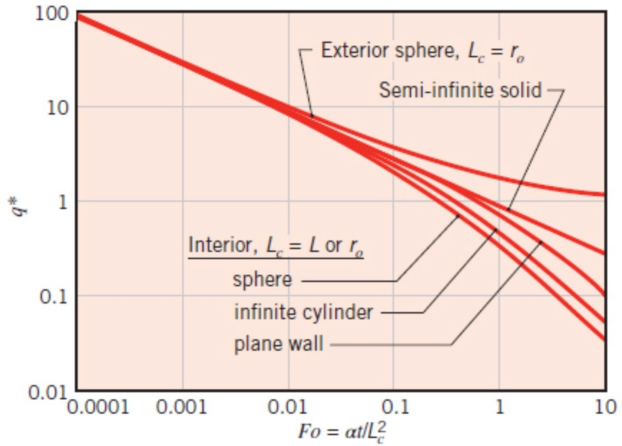


Figure 5.10

Case 3 Surface Convection

Semi-Infinite Solid - Solution

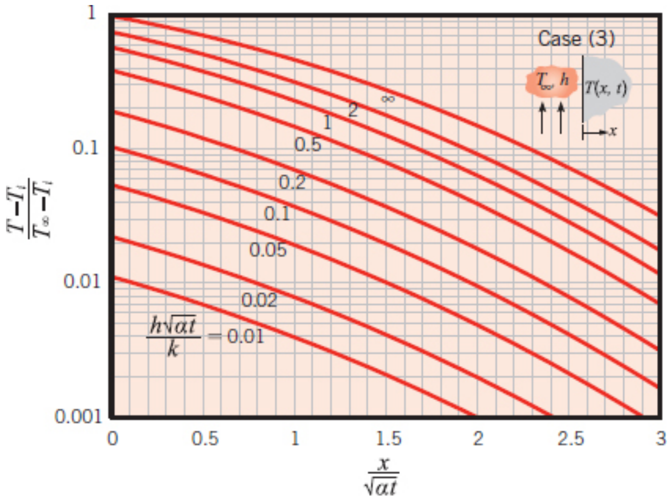


FIGURE 5.8 Temperature histories in a semi-infinite solid with surface convection.

Example 5.7