7C - 2ND CLASS

$$T(\eta) = T_s + A_2 \int_0^{\eta} e^{-\eta^2}\, d\eta$$

where $\int_0^{\eta} e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2} \operatorname{erf}(\eta)$

error function
$\operatorname{erf}(0) = 0$
$\operatorname{erf}(\infty) = 1$

$\operatorname{erfc}(\eta) = 1 - \operatorname{erf}(\eta)$

$$T(\eta) = T_s + A_2 \frac{\sqrt{\pi}}{2} \operatorname{erf}(\eta)$$

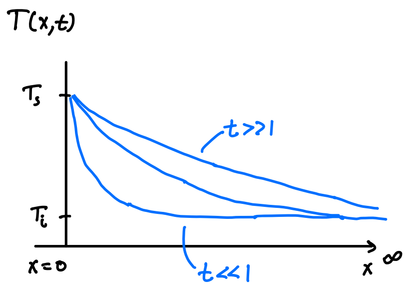
use B.C. at $\eta = \infty$,
$T(\eta = \infty) = T_i$
$T_i = T_s + \frac{A_2 \sqrt{\pi}}{2}(1)$
$A_2 = \frac{2}{\sqrt{\pi}}(T_i - T_s)$

So,

$$\boxed{T(\eta) = T_s + (T_i - T_s)\operatorname{erf}(\eta)}$$

or,

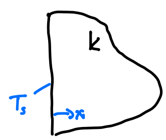
$$T(x,t) = T_s + (T_i - T_s)\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$



Whats the heat flux?

$$q_s''(x,t) = ?$$
$$= -k \left.\frac{\partial T}{\partial x}\right|_{x=0}$$
$$= -k \left.\frac{dT}{d\eta}\frac{\partial \eta}{\partial x}\right|_{\eta=0}$$
$$= -\frac{k}{2\sqrt{\alpha t}} \boxed{\left.\frac{dT}{d\eta}\right|_{\eta=0}}$$

?



$\rightarrow \frac{dT}{d\eta} = ?$

$$\frac{dT}{d\eta} = (T_i - T_s)\frac{d}{d\eta}(\operatorname{erf}(\eta))$$

$$\frac{d}{d\eta}(\operatorname{erf}(\eta)) =$$

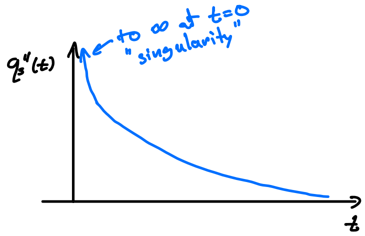
$$\frac{d}{d\eta}\left[\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2} d\eta\right]$$

$$= \frac{2}{\sqrt{\pi}} e^{-\eta^2} \Rightarrow \left.\frac{dT}{d\eta}\right|_{\eta=0} = (T_i - T_s)\frac{2}{\sqrt{\pi}}$$

So,

$$q_s''(t) = \frac{-k}{2\sqrt{\alpha t}}(T_i - T_s)\frac{2}{\sqrt{\pi}}$$

$$\boxed{q_s''(t) = \frac{k(T_s - T_i)}{\sqrt{\pi\alpha}}\, t^{-1/2}}$$

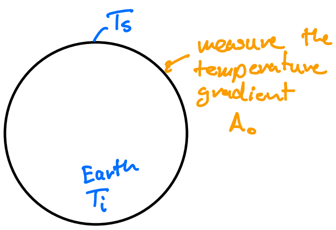


GRAPH TAKE AWAYS:

1. At $t=0$ $q_s'' \to \infty$ due to non-physical, instantaneous change of T_i to T_s at $x=0$.
2. We see $q_s'' \sim 1/\sqrt{t}$ ["goes like"]

EXAMPLE

Determine the age of the earth



Earth is cooling

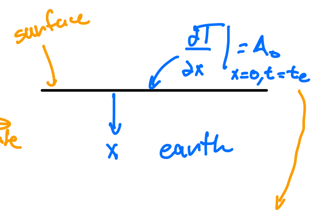
$T_i = 4000\,°C$

$T_s = 0\,°C$

$\alpha = 1\times10^{-6}\ m^2/s$

$A_o = 0.03\ °C/m$

apply semi-infinite soln



Find a symbolic solution for t_e?

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$$

$T(0,t) = T_s$

$T(\infty,t) = T_i$

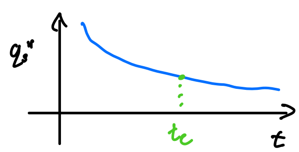
$T(x,0) = T_i$

similarity transform

$$T(x,t) = T_s + (T_i - T_s)\,\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

$$q_s''(t) = \frac{k(T_s - T_i)}{\sqrt{\pi\alpha}}\, t^{-1/2}$$

$t_e = ?$



$$q_s''(t=t_e) = -k\left.\frac{\partial T}{\partial x}\right|_{x=0,t=t_e} = -kA_o = \frac{k(T_s - T_i)}{\sqrt{\pi\alpha}}\, t_e^{-1/2}$$

$$\Rightarrow \boxed{t_e = \frac{(T_s - T_i)^2}{A_o^2\,\pi\,\alpha}} \longrightarrow \boxed{t_e = 179 \text{ million years}}$$

– Source: Kelvin (1890's)